Learning Together Through Collaborative Research: The Case of Proof in Secondary Mathematics

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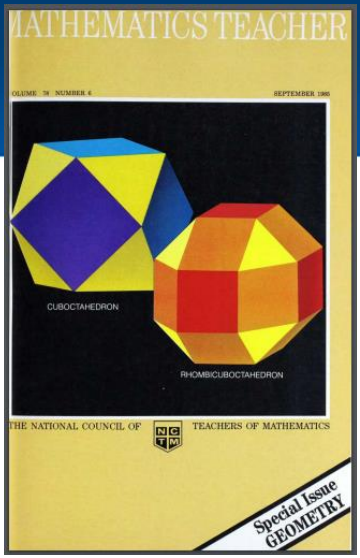
Research on Proof in School Mathematics

- Proof is important –
  the "guts of mathematics" (Wu, 1996).

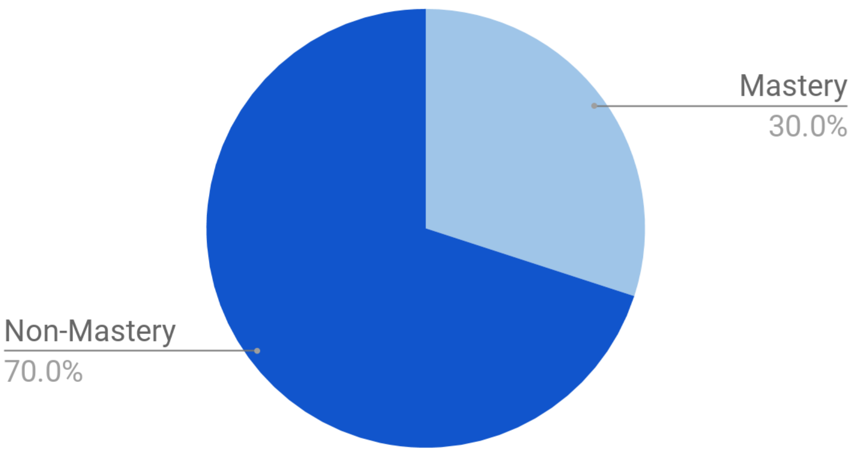
  BUT

- Proof is challenging for teachers to teach (e.g., Knuth, 2002, Cirillo, 2009; 2014).
- Proof is difficult for students to learn (Senk, 1985; McCrone & Martin, 2004).

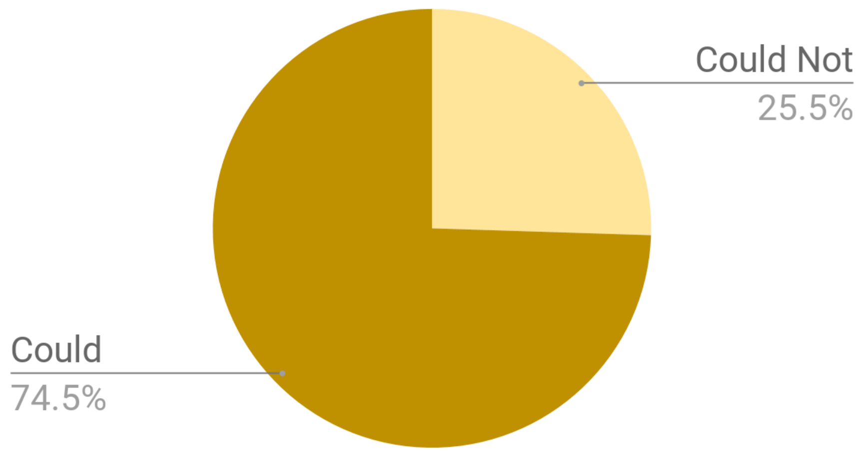
How well do students write geometry proofs?



Reached 75% Mastery of Proof



Ability to Write At Least 1 Valid Proof



Sharon Senk (1985)

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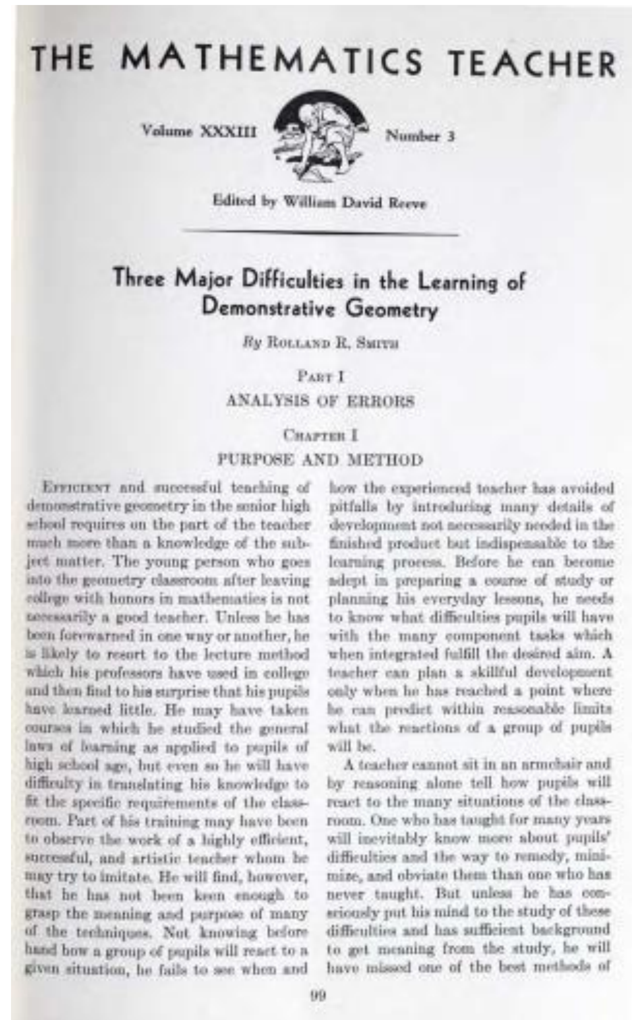
Senk's Recommendations

We must immediately look for more effective ways to teach proof in geometry. We should:

- Pay special attention to teaching students to *start* a chain of reasoning;
- Place greater emphasis on the meaning of proof than we do currently; and
- Teach students how, why, and when they can transform a diagram in a proof.

Three Major Difficulties in the Learning of Demonstrative Geometry

Rolland R. Smith (1940)



THE MATHEMATICS TEACHER

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Three Major Difficulties in the Learning of Demonstrative Geometry

By Rolland R. Smith

Part I

ANALYSIS OF ERRORS

Chapter I

PURPOSE AND METHOD

Efficient and successful teaching of demonstrative geometry in the senior high school requires on the part of the teacher much more than a knowledge of the subject matter. The young person who goes into the geometry classroom after leaving college with honors in mathematics is not necessarily a good teacher. Unless he has been forewarned in one way or another, he is likely to resort to the lecture method which his professors have used in college and then find to his surprise that his pupils have learned little. He may have taken courses in which he studied the general laws of learning as applied to pupils of high school age, but even so he will have difficulty in translating his knowledge to fit the specific requirements of the classroom. Part of his training may have been to observe the work of a highly efficient, successful, and artistic teacher whom he may try to imitate. He will find, however, that he has not been keen enough to grasp the meaning and purpose of many of the techniques. Not knowing beforehand how a group of pupils will react to a given situation, he fails to see when and how the experienced teacher has avoided pitfalls by introducing many details of development not necessarily needed in the finished product but indispensable to the learning process. Before he can become adept in preparing a course of study or planning his everyday lessons, he needs to know what difficulties pupils will have with the many component tasks which when integrated fulfill the desired aim. A teacher can plan a skillful development only when he has reached a point where he can predict within reasonable limits what the reactions of a group of pupils will be.

A teacher cannot sit in an armchair and by reasoning alone tell how pupils will react to the many situations of the classroom. One who has taught for many years will inevitably know more about pupils' difficulties and the way to remedy, minimize, and obviate them than one who has never taught. But unless he has consciously put his mind to the study of these difficulties and has sufficient background to get meaning from the study, he will have missed one of the best methods of

"Three Serious Learning Difficulties"

- Lack of familiarity with geometric figures
- Not sensing the meaning of the if-then relationship
- Inadequate understanding of the meaning of proof

Students' Difficulties with Proof in Geometry

- "In summary, we have seen that students are extremely unsuccessful with formal proof in geometry."

(Clements & Battista, 1992)

- "The teaching of mathematical proof appears to be a failure in almost all countries."

(Hershkowitz et al., 2002, p. 675)

Calls for Additional Research…

- “The mandate to involve students in proving is likely to be met with the development of tools and norms that teachers can use to enable students to prove and to demonstrate that they are indeed proving.”

  (Herbst, 2002, p. 200)

- “…research is needed to understand the conditions in which teachers work and how those conditions impact the mathematical work that teachers can sustain”

  (Herbst, 2006, p. 314)

Timeline of Progress

Smith (1940)

↓

Senk (1985)

↓

Cirillo (2020)

Three Studies

- 2005-2008: Longitudinal Dissertation Study
- 2010-2013: The Geometry Proof Project
- 2015-2020: Proof in Secondary Classrooms: Decomposing a Central Mathematical Practice (i.e., The PISC Project)

STUDY 1: THE CASE OF MATT

Matt

You can't teach somebody how to do a proof….I mean if a student's really gonna do a mathematical proof, you look at the problem and you either see how you do it or you don't.

Textbook Examples

- Reasoning with Properties from Algebra

EXAMPLE 2 *Writing Reasons*

Solve $55z - 3(9z + 12) = -64$ and write a reason for each step.

SOLUTION

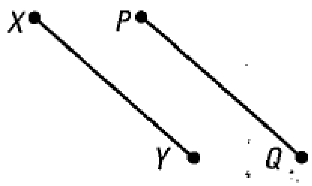
$55z - 3(9z + 12) = -64$	Given
$55z - 27z - 36 = -64$	Distributive property
$28z - 36 = -64$	Simplify.
$28z = -28$	Addition property of equality
$z = -1$	Division property of equality

Textbook Examples

- Proving Statements about Segments

EXAMPLE 1 ***Symmetric Property of Segment Congruence***

You can prove the Symmetric Property of Segment Congruence as follows.



GIVEN ▶ $\overline{PQ} \cong \overline{XY}$

PROVE ▶ $\overline{XY} \cong \overline{PQ}$

Statements	Reasons
1. $\overline{PQ} \cong \overline{XY}$	**1.** Given
2. $PQ = XY$	**2.** Definition of congruent segments
3. $XY = PQ$	**3.** Symmetric property of equality
4. $\overline{XY} \cong \overline{PQ}$	**4.** Definition of congruent segments

The Case of Matt: Overall Findings

- Despite strong content knowledge and a good teacher prep program, Matt was at a loss for teaching proof beyond show-and-tell.
- Matt wanted to teach “real math,” not just show students completed Theorems in the boxes in his textbook.
- Matt’s focus shifted from getting through the required theorems to attempting to teach students to prove.

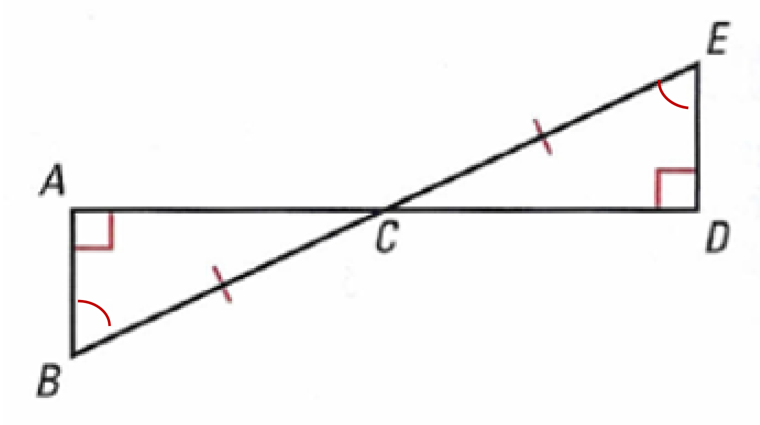
STUDY 2: THE CASE OF MIKE

Mike, High School Geometry Teacher

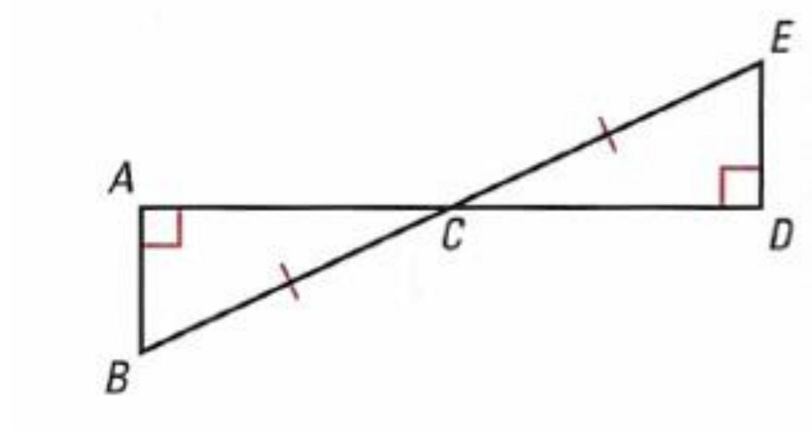
- 8 years of experience at start of project
- Mathematics and Science background
- Conventional Prentice Hall *Geometry* textbook
- Private boys' school
- Described students as motivated, curious, confident, intelligent, and affluent

Mike Began Proof with Triangle Congruence

1. GIVEN: $\angle A \cong \angle D$, $\angle B \cong \angle E$
$\overline{BC} \cong \overline{EC}$
PROVE: $\Delta ABC \cong \Delta DEC$



20. GIVEN ▶ $\overline{AB} \perp \overline{AD}$, $\overline{DE} \perp \overline{AD}$,
$\overline{BC} \cong \overline{EC}$
PROVE ▶ $\triangle ABC \cong \triangle DEC$



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BACK TO MATT FOR A BRIEF MOMENT…

Matt – Year 2

- "On Friday the students will begin constructing their own deductive proofs. Unfortunately, there is no good way, in my opinion, to 'teach' proofs. Students simply have to do them – like learning to swim by drowning."

- "Ok, there's only so many of these that I can do with us together. I just kind of, got to keep throwing you in the deep end. Letting you thrash around for awhile. And then throw you a floaty. Haul you back out and then throw you back in. Alright?"

(Cirillo, 2008)

BACK TO MIKE…

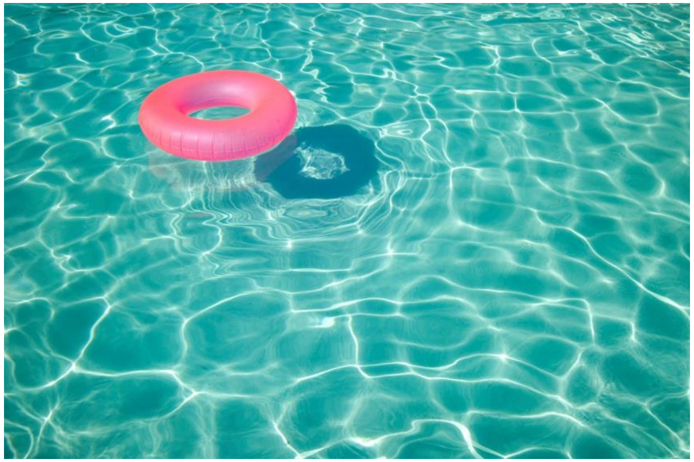
Things I need to know:

- How do I know what steps to write?
- How do I know what order the steps are in?
- Argh! I don't even know where to start!!!
- How big should I make the T?
- What reasons am I allowed to use?
- How many steps do I need to write?

What makes teaching proof in geometry so tough?

- Curriculum
- Student Readiness
- Lack of recommendations for scaffolding the introduction to proof (i.e., understanding of the "shallow end" of the proof pool)

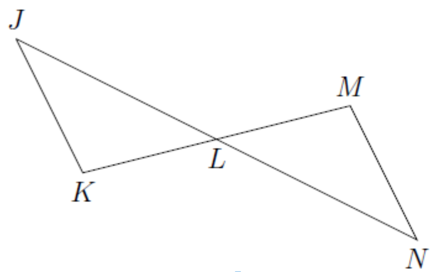
- What is going on for *students* when we introduce proof?



Perpendicular lines intersect to form right angles.

Given: $\overline{JN}$ bisects $\overline{KM}$
$\overline{JK} \perp \overline{KM}$
$\overline{MN} \perp \overline{KM}$

Prove: $\angle KJL \cong \angle MNL$



Statements | Reasons

No Given? What can we assume from a diagram?

$\overline{JK} \perp \overline{KM}$ and $\overline{MN} \perp \overline{KM}$
(Given)

$\angle K$ and $\angle M$ are right angles
(Definition of Perpendicular Lines)

$\angle K \cong \angle M$
(Theorem: If two angles are right angles, then they are congruent.)

$\overline{JN}$ bisects $\overline{KM}$
(Given)

L is the midpoint of $\overline{KM}$
(Definition of Line Segment Bisector)

$\overline{KL} \cong \overline{LM}$
(Definition of Midpoint)

$\angle JLK$ and $\angle NLM$ are vertical angles
(Definition of Vertical Angles)

$\angle JLK \cong \angle NLM$
(Theorem: If two angles are vertical angles, then they are congruent.)

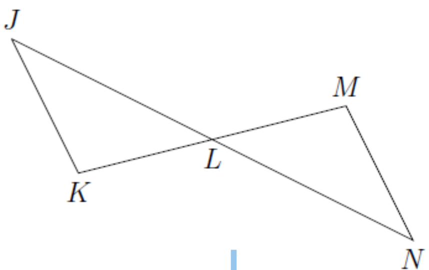
$\triangle KJL \cong \triangle MNL$
(ASA $\cong$ ASA)

$\angle KJL \cong \angle MNL$
(CPCTC)

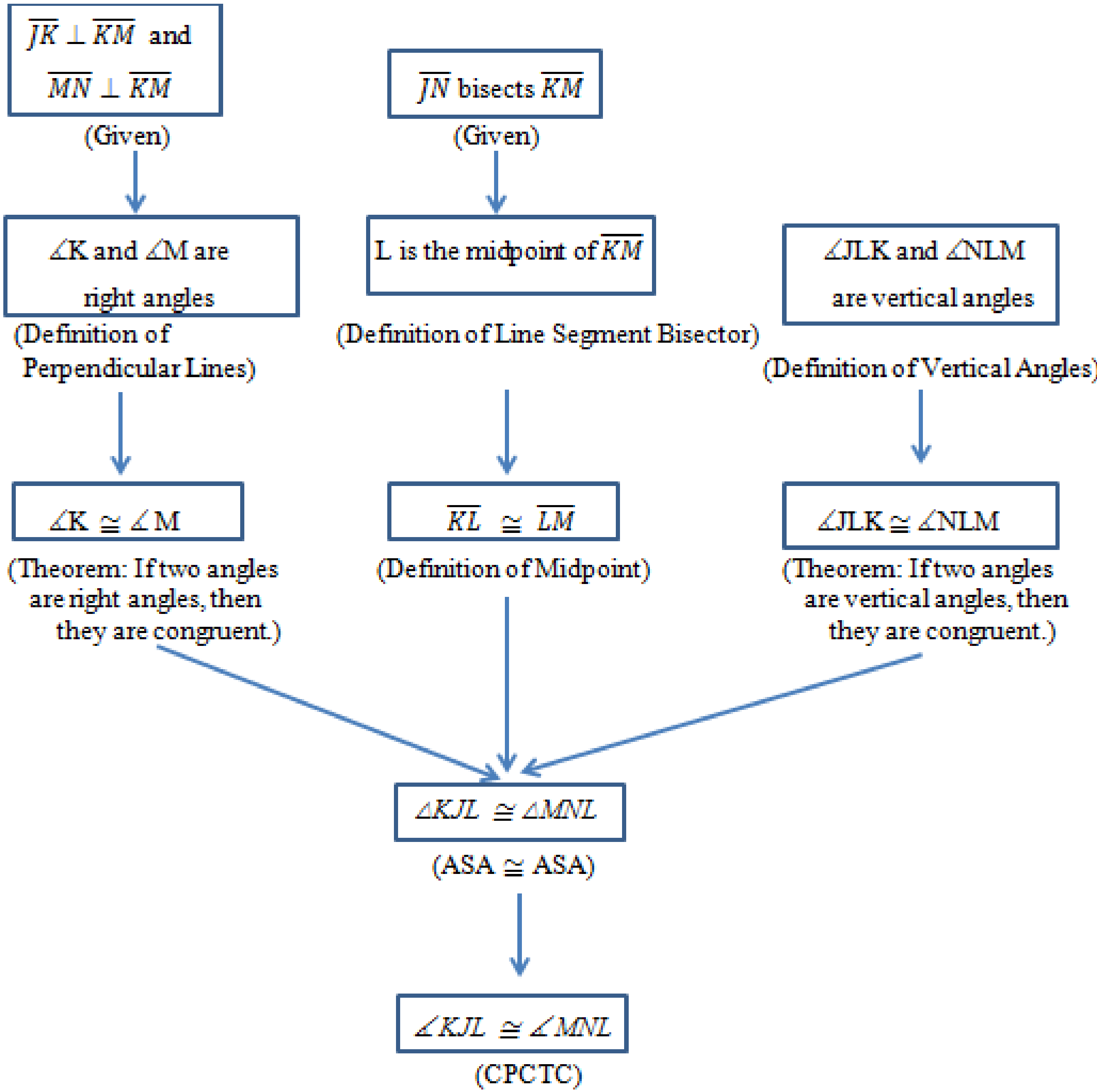
A Proof

Given: $\overline{JN}$ bisects $\overline{KM}$
$\overline{JK} \perp \overline{KM}$
$\overline{MN} \perp \overline{KM}$

Prove: $\angle KJL \cong \angle MNL$



Statements	Reasons



If there was a shallow end to teaching proof, what would it look like?

STUDY 3: THE PISC PROJECT

PISC Project Timeline

CAMP

NOW

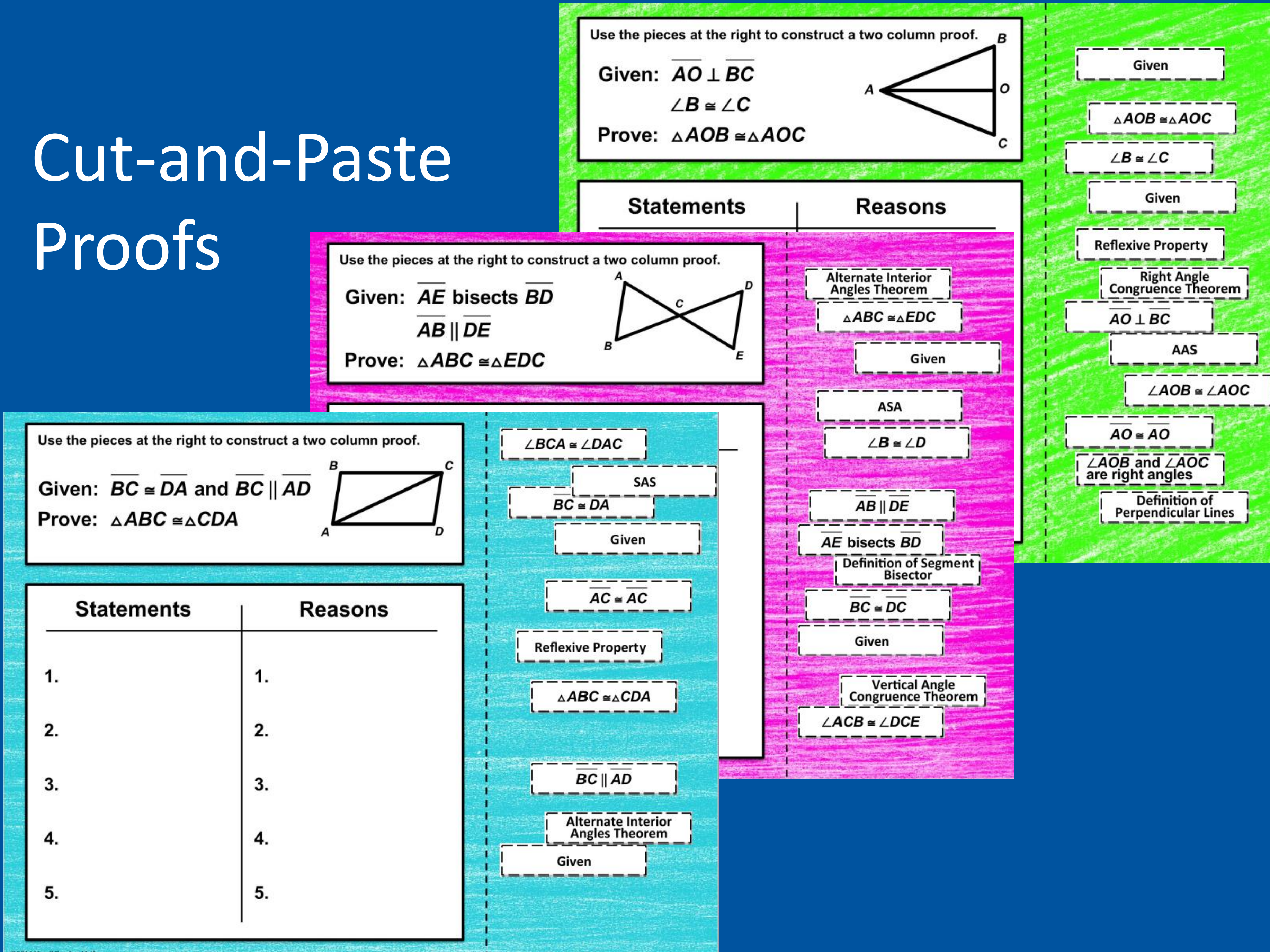
Intervention

Planning Year	Baseline Data Collection & Lesson Piloting	Professional Development & Summer Lesson Study	Pilot Lessons (Core Teachers)	Pilot Lessons (Again) (Core Teachers)	Publication & Dissemination
Phase I	Phase II	Phase III	Phase IV	Phase V	Phase VI
2015-2016	2016-2017	Spring & Summer, 2017	2017-2018	2018-2019	2019-2020

Used Senk & Usiskin's assessments with Control and Experimental Groups

SOME CLASSROOM VIDEOS
Y1 → Y3 (ACTUALLY Y2, Y4 OF PISC)

Cut-and-Paste Proofs



Year 1: First Day of Triangle Congruence Proof

What do you notice and wonder?

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Year 1: First Day of Triangle Congruence Proof

What do you notice and wonder?

"I noticed a lot of really great things you guys were doing. You remembered to put your Given information first and to put what you're trying to prove last and for the most part it looked like we had a lot of things in the correct order, but some of you, I feel like just put them there because you knew they had to be there, but you didn't really go through the steps in the correct kind of order. So that's what we're going to work on today."

Year 1: First Day of Triangle Congruence Proof

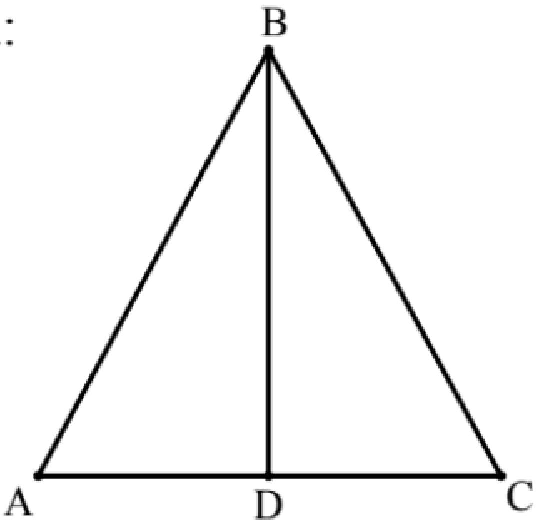
- No logic
- 52 minutes
- Unsure
  - “They said…”

Year 3: First Day of Triangle Congruence Proof

Given: $\overline{BD}$ is the ⊥ bisector of $\overline{AC}$

Prove: $\triangle ABD \cong \triangle CBD$

Diagram:



Year 3: First Day of Triangle Congruence Proof

- Presentation of student work
- Modeling how to discuss and critique the reasoning of others

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Year 3: First Day of Triangle Congruence Proof

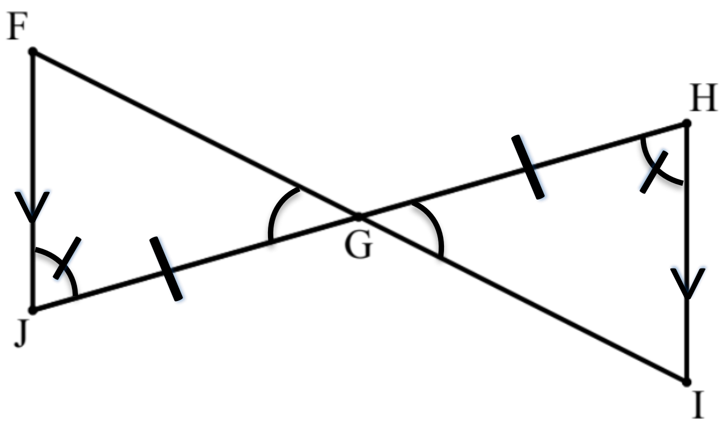
What do you notice and wonder?

Year 3: First Day of Triangle Congruence Proof

- Confidence in content
- Student input on making the proof better
- Using true logic

Given: $\overline{FJ} \parallel \overline{HI}$
$\overline{FI}$ bisects $\overline{JH}$ at G

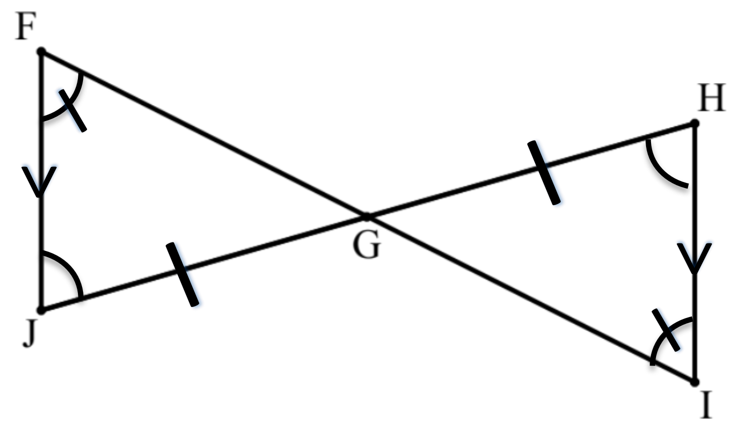
Prove: $\triangle JFG \cong \triangle HIG$



$$\Delta JFG \cong \Delta HIG$$
$$ASA \cong ASA$$

Given: $\overline{FJ} \parallel \overline{HI}$
$\overline{FI}$ bisects $\overline{JH}$ at G

Prove: $\triangle JFG \cong \triangle HIG$



$$\Delta JFG \cong \Delta HIG$$
$$AAS \cong AAS$$

Year 3: First Day of Triangle Congruence Proof

- Student presentation
- Various outcomes
- Creating opportunities for students to engage with another's reasoning

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Year 3: First Day of Triangle Congruence Proof

What do you notice and wonder?

Year 3: First Day of Triangle Congruence Proof

- Student-focused
- Various methods to solve
- Teacher discourse moves

PISC Project Timeline

Video 1

WHAT HAPPENED IN HERE?

Videos 2 & 3

Planning Year	*Baseline Data Collection & Lesson Piloting*	*Professional Development & Summer Lesson Study*	*Pilot Lessons (Core Teachers)*	*Pilot Lessons (Again) (Core Teachers)*	*Publication & Dissemination*
Phase I	**Phase II**	**Phase III**	**Phase IV**	**Phase V**	**Phase VI**
2015-2016	2016-2017	Spring & Summer, 2017	2017-2018	2018-2019	2019-2020

WHAT HAPPENED?

Between Year 1 and 3: What happened?

- Professional Development
  - Student Thinking
  - Summer Camp
    - Lesson Study
    - Debriefing
- Lessons and readings on Teacher Discourse Moves
- Teaching the Lessons
  - Video Recorded
  - Feedback
  - Daily Reflections
- Continuous PD
  - Met as a Group – Improved Lessons
  - Control Group

WHAT HAPPENED?

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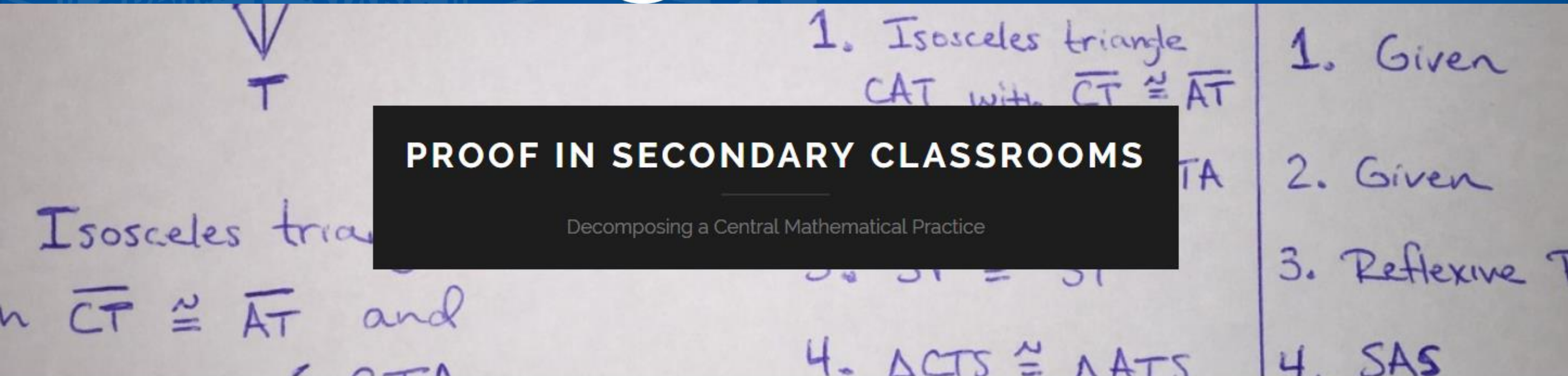


PROOF IN SECONDARY CLASSROOMS

Decomposing a Central Mathematical Practice

Welcome!

The *Proof in Secondary Classrooms (PISC)* project is a five-year CAREER grant funded by the National Science Foundation (PI: Michelle Cirillo). PISC will develop an intervention to support the teaching and learning of proof in the context of geometry. This study takes as its premise that if we introduce proof, by first teaching students particular sub-goals of proof, such as how to draw a conclusion from a given statement and a definition, then students will be more successful with constructing proofs on their own.

PISC will draw on findings and artifacts from a previous 3-year study, funded by the Knowles Science Teaching Foundation, which considered the challenges of teaching proof in geometry. In this earlier study, classroom and interview data

LINKS

Professor Cirillo's Homepage
UD Department of Mathematical Sciences
UD Math Education PhD Program
UD Homepage

RESEARCH FUNDED BY:

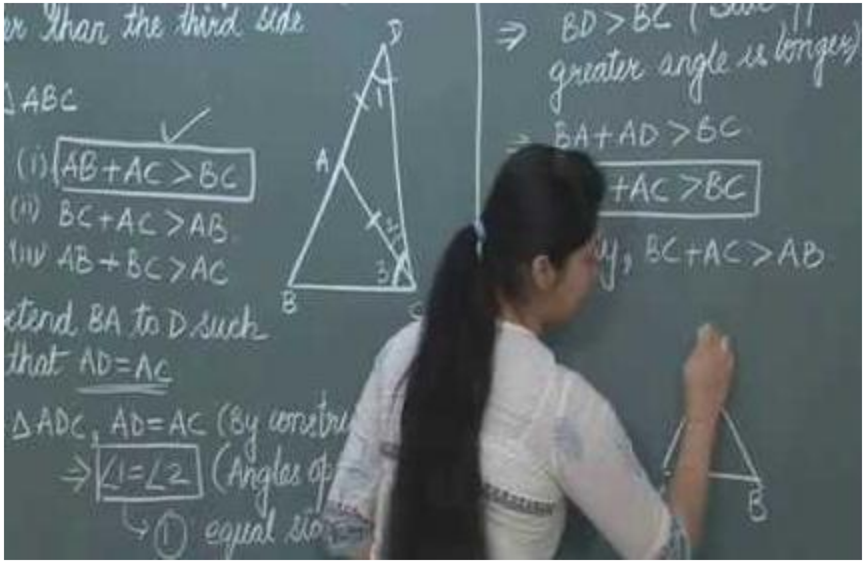
Decomposing Proof

Geometry Proof Scaffold: A Pedagogical Framework for Teaching Proof

Sub-Goals	Descriptions	Competencies
Understanding Geometric Concepts	This sub-goal highlights the importance of understanding the building blocks of geometry.	1) Having accurate "mental pictures" of geometric concepts (i.e., having a concept image)
		2) Being able to verbally describe geometric concepts, ideally being fluent with one or more definitions of the concept (i.e., having or developing a concept definition)
		3) Determining examples and non-examples
		4) Understanding connections between classes of geometric objects, where they overlap, and how they are contained within other classes (i.e., understanding mathematical hierarchy)
Coordinating Geometric Modalities	This sub-goal highlights the ways in which the mathematics register draws on a range of modalities.	1) Translating between language and diagram
		2) Translating between diagram and symbolic notation
		3) Translating between language and symbolic notation
Defining	This sub-goal highlights the nature of definitions, their logical structure, how they are written, and how they are used.	1) Writing a "good" definition (includes necessary and sufficient properties)
		2) Knowing definitions are not unique (i.e., geometric objects can have different definitions)
		3) Understanding how to write and use definitions as biconditionals
Conjecturing	This sub-goal recognizes that conjecturing is an important part of mathematics and proving.	1) Understanding that empirical reasoning can be used to develop a conjecture but that it is not sufficient proof of the conjecture
		2) Being able to turn a conjecture into a testable conditional statement.
		3) Seeking out counterexamples to test conjectures and knowing that only one counterexample is needed to disprove a conjecture
		4) Understanding that when testing a conjecture, you are testing it for every case so you might begin by writing: "All," "Every, or "For any"
Drawing Conclusions	This sub-goal presents the idea of an open-ended task that leads to conclusions that can be drawn from given statements and/or a diagram.	1) Understanding what can and cannot be assumed from a diagram
		2) Knowing when and how definitions and/or "Given" information can be used to draw a conclusion from a statement about a mathematical object
		3) Using postulates, definitions, and theorems (or combinations of these) to draw valid conclusions from some given information
Understanding Common Sub-arguments	This sub-goal recognizes that there are common short sequences of statements and reasons that are used frequently in proofs and that these pieces may appear relatively unchanged from one proof to the next.	1) Recognizing a sub-argument as a branch of proof and how it fits into the larger proof
		2) Understanding what valid conclusions can be drawn from a given statement and how those make a sub-argument (i.e., knowing some commonly occurring sub-arguments)
		3) Understanding how to write a sub-argument using notation and acceptable language (where "acceptable" is typically determined by the teacher)
Understanding Theorems	This sub-goal highlights the nature of theorems, their logical structure, how they are written, and how they are used.	1) Interpreting a theorem statement to determine the hypothesis and conclusion, and, if needed, providing an appropriate diagram
		2) If applicable, marking a diagram that satisfies the hypothesis of a proof
		3) Rewriting a theorem written in words in symbols and vice versa
		4) Understanding that a theorem is not a theorem until it has been proven
		5) Understanding that one cannot use the conclusions of the theorem itself to prove the conclusions of that theorem (i.e., avoiding circular reasoning)
		6) Understanding that theorems are mathematical statements that are only sometimes biconditionals
		7) Understanding the connection between logic and a theorem, for example, how to write the contrapositive of a conditional statement
Understanding the Nature of Proof	This sub-goal highlights the nature of proof, proof structure, and how the laws of logic are applied.	1) Understanding that the only way to sanction the truth of a conjecture is through deductive proof (rather than empirical reasoning)
		2) Exploring a pathway for constructing a proof (i.e., the problem solving aspect of proving)
		3) Understanding that proofs are constructed using axioms, postulates, definitions, and theorems and that they follow the laws of logic
		4) Knowing what language is acceptable to use and how to write up a proof
		5) Recognizing that if you prove that something is true for one particular geometric object, then it is true for all of them

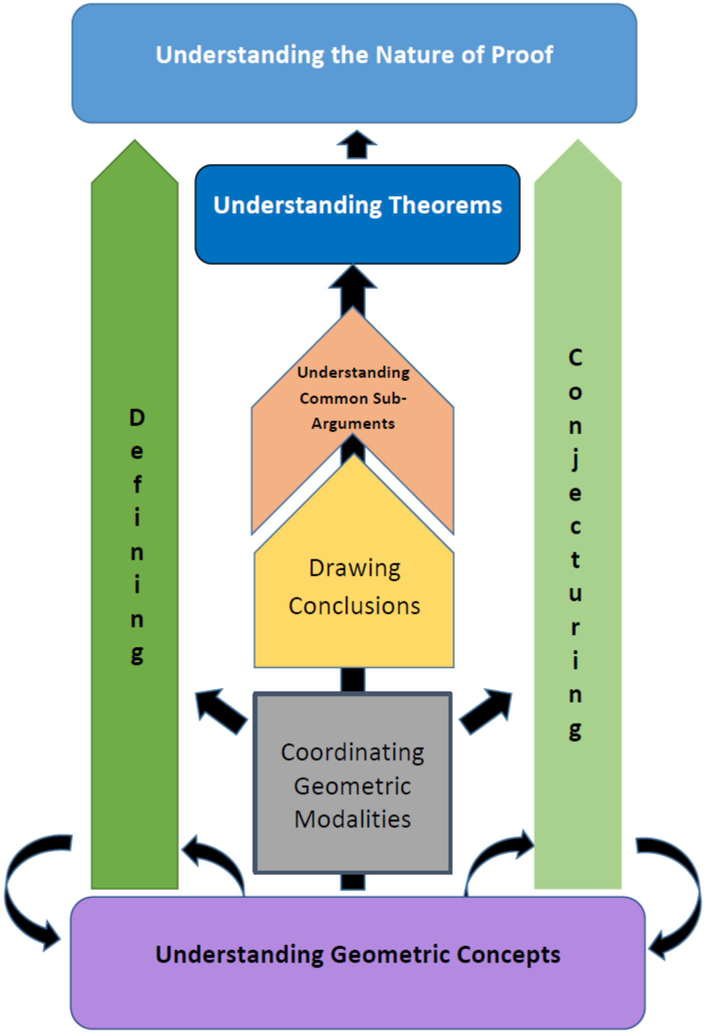
Show-and-Tell vs. An On-Ramp

- Teacher Show-and-Tell
- On-Ramp

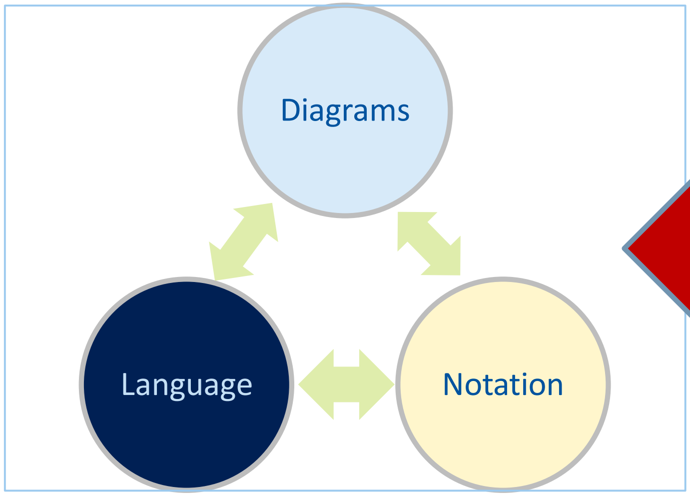


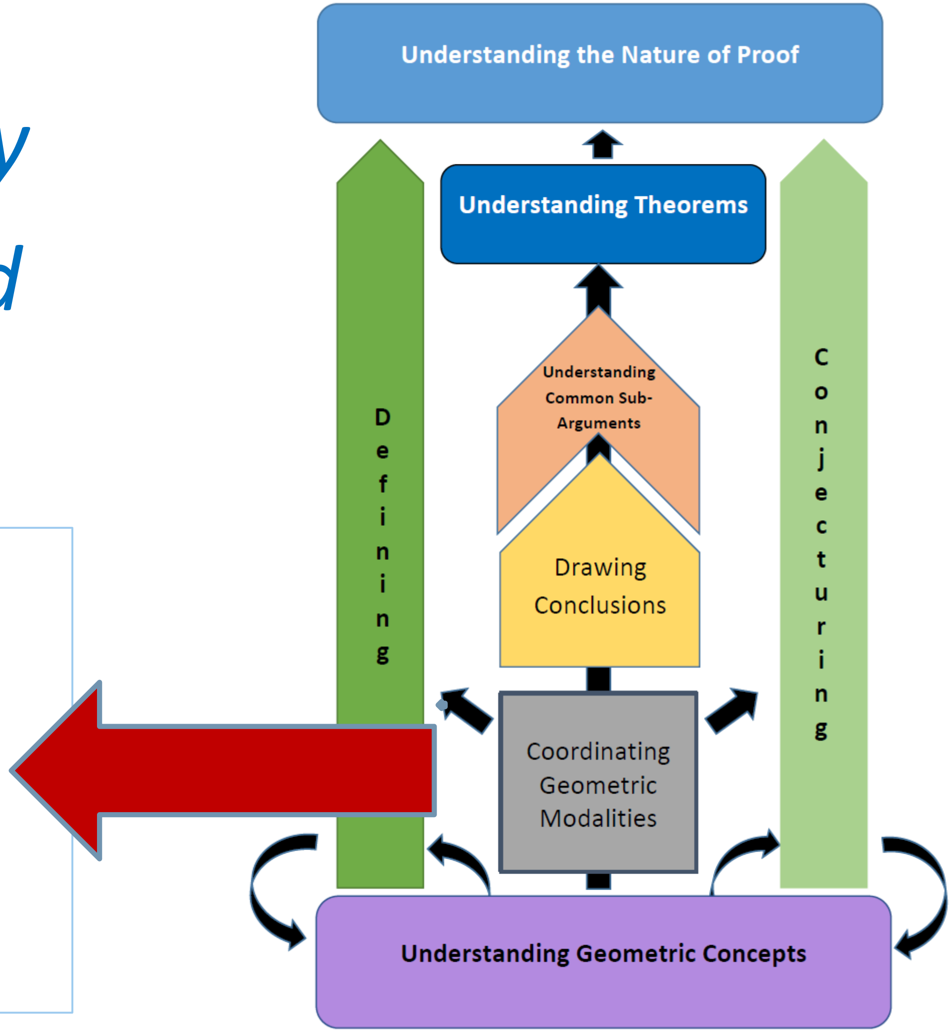
The Geometry Proof Scaffold

(i.e., the "GPS")



The Geometry Proof Scaffold





The Geometry Proof Scaffold

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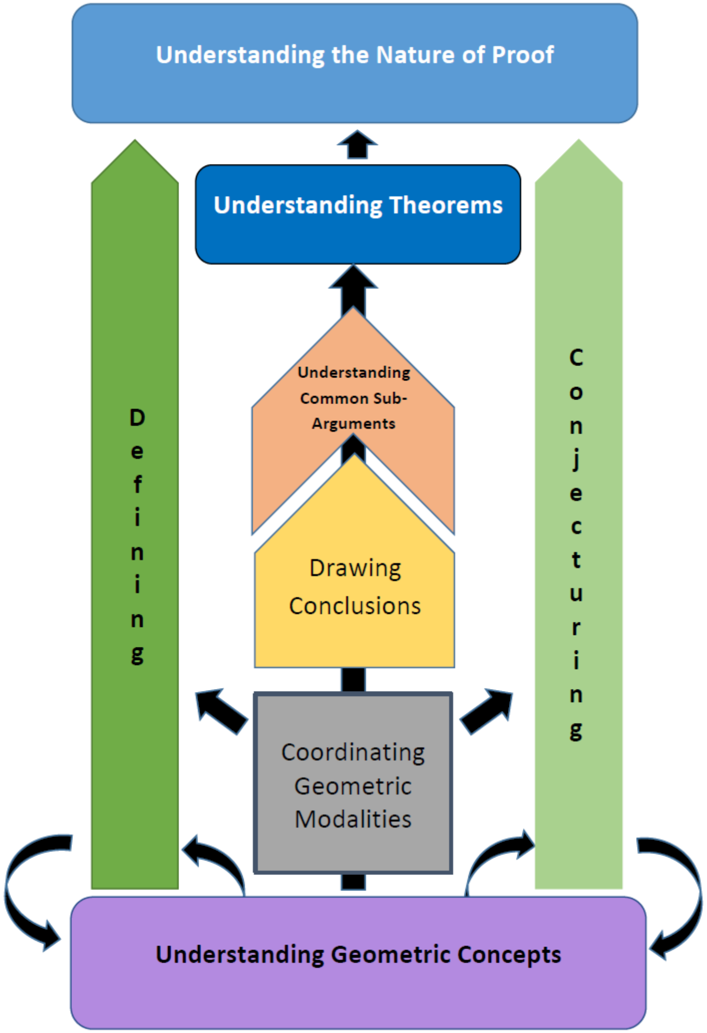
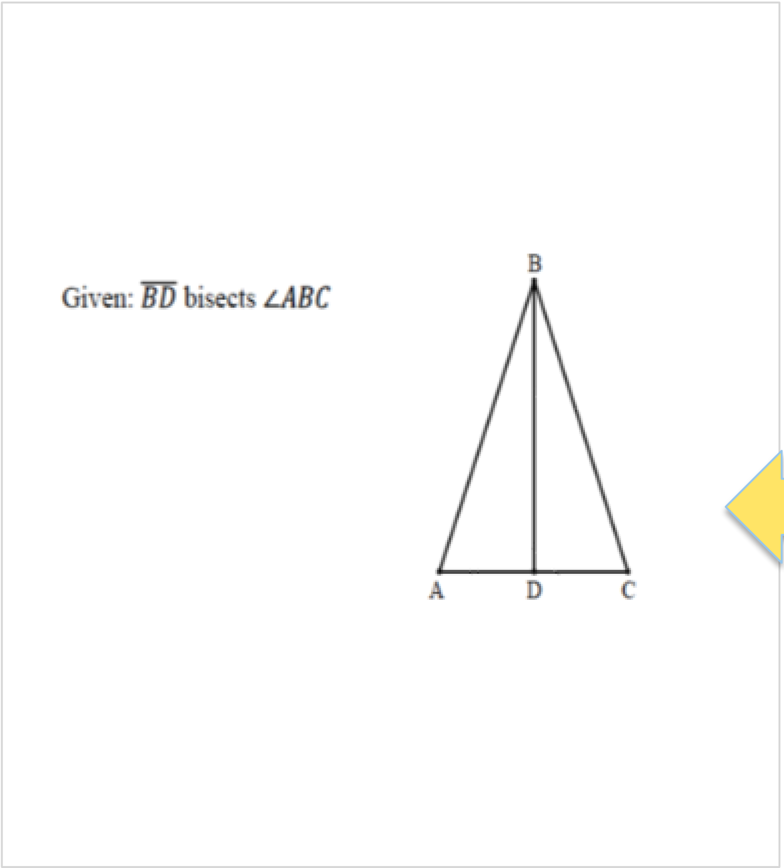
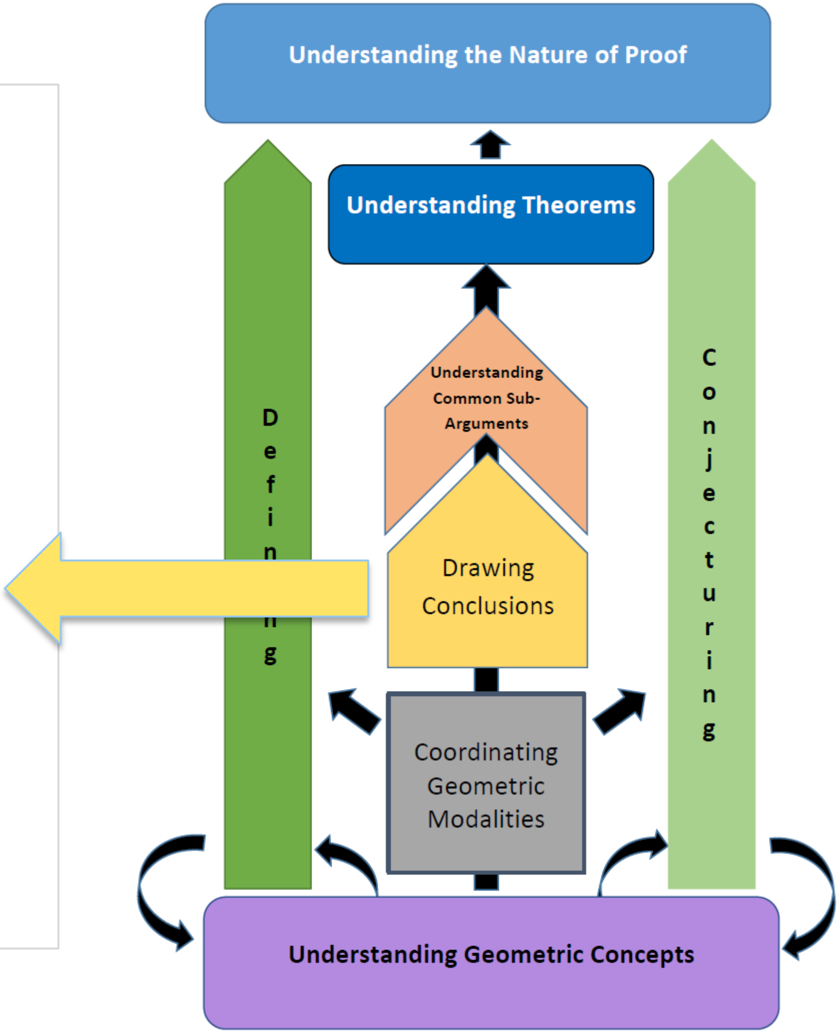
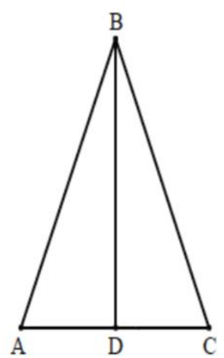



Given: $\overline{BD}$ bisects $\angle ABC$
B
A
D
C

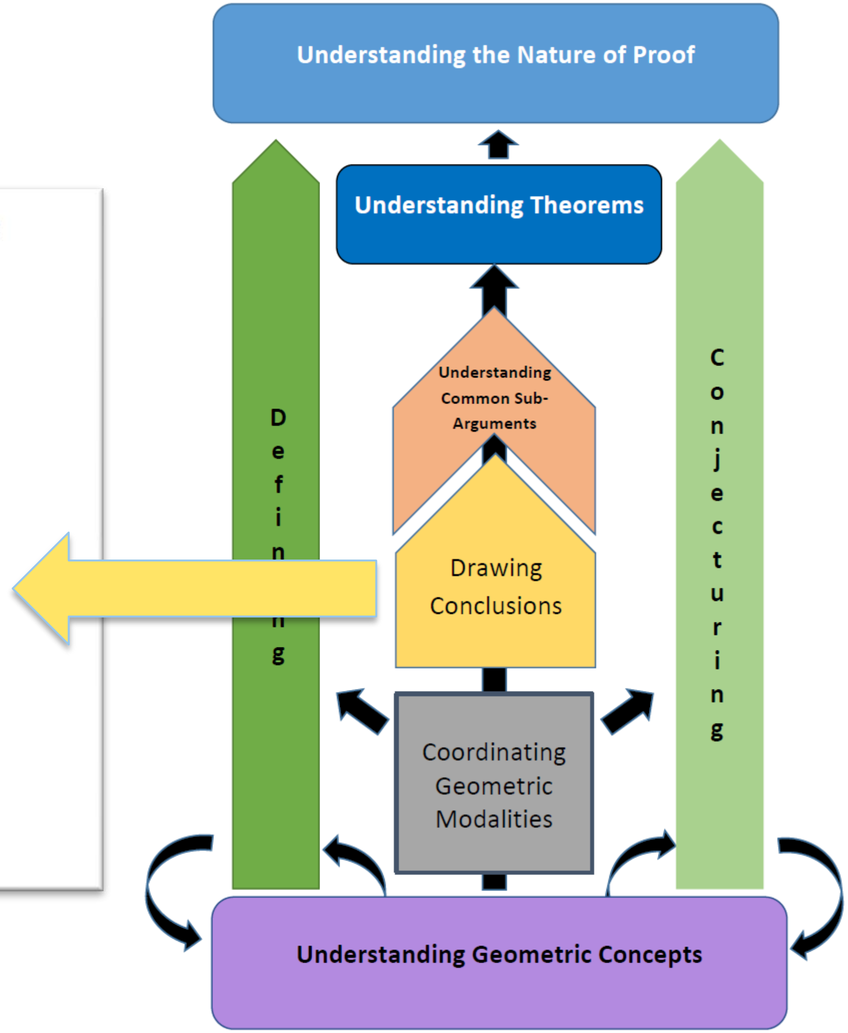

Understanding the Nature of Proof
Understanding Theorems
Understanding Common Sub-Arguments
Drawing Conclusions
Coordinating Geometric Modalities
Defining
Conjecturing
Understanding Geometric Concepts

2. Which of the following statements could you conclude from the Given information and the figure?

Given: $\overline{BD}$ bisects $\angle ABC$

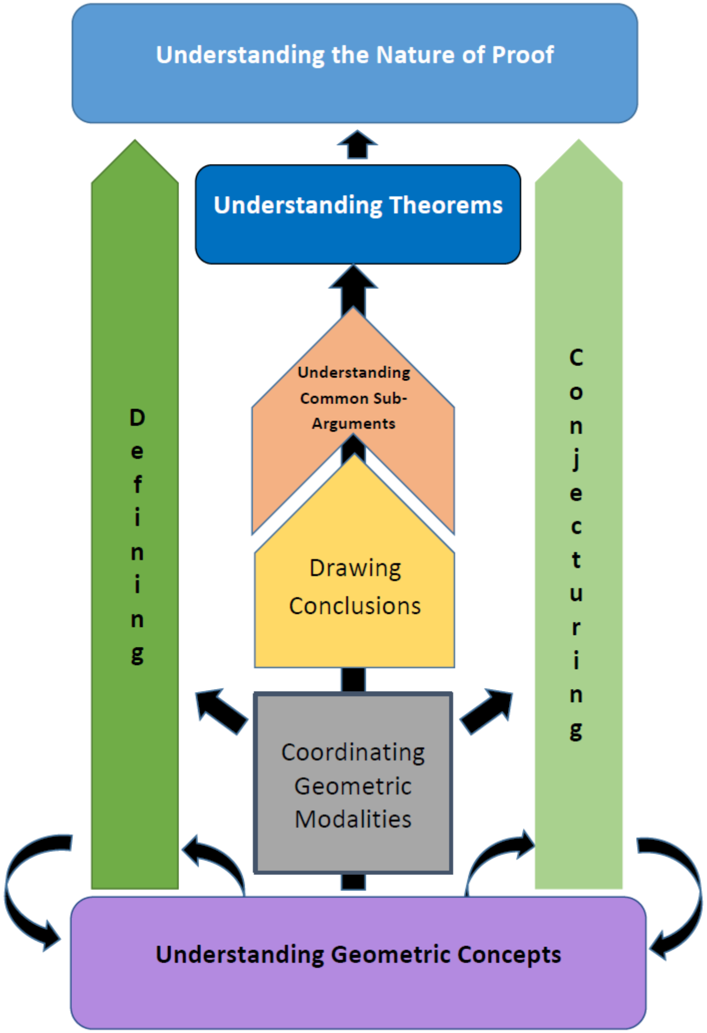


A. $\overline{BD} \perp \overline{AC}$
B. $\overline{AD} \cong \overline{CD}$
C. $\angle ABD \cong \angle CBD$
D. $\triangle$ABC is Isosceles
E. All of the above



The Geometry Proof Scaffold

(i.e., the "GPS")



PISC Research Questions

- How do teachers *introduce* proof in geometry?
- When engaging in lesson study based on introducing proof by first teaching particular sub-goals of proof, how do teacher respond to and execute the lesson plans?
- How do students respond to these lessons?
- How do students in the control and experimental groups think about proof and perform on a set of proof tasks?

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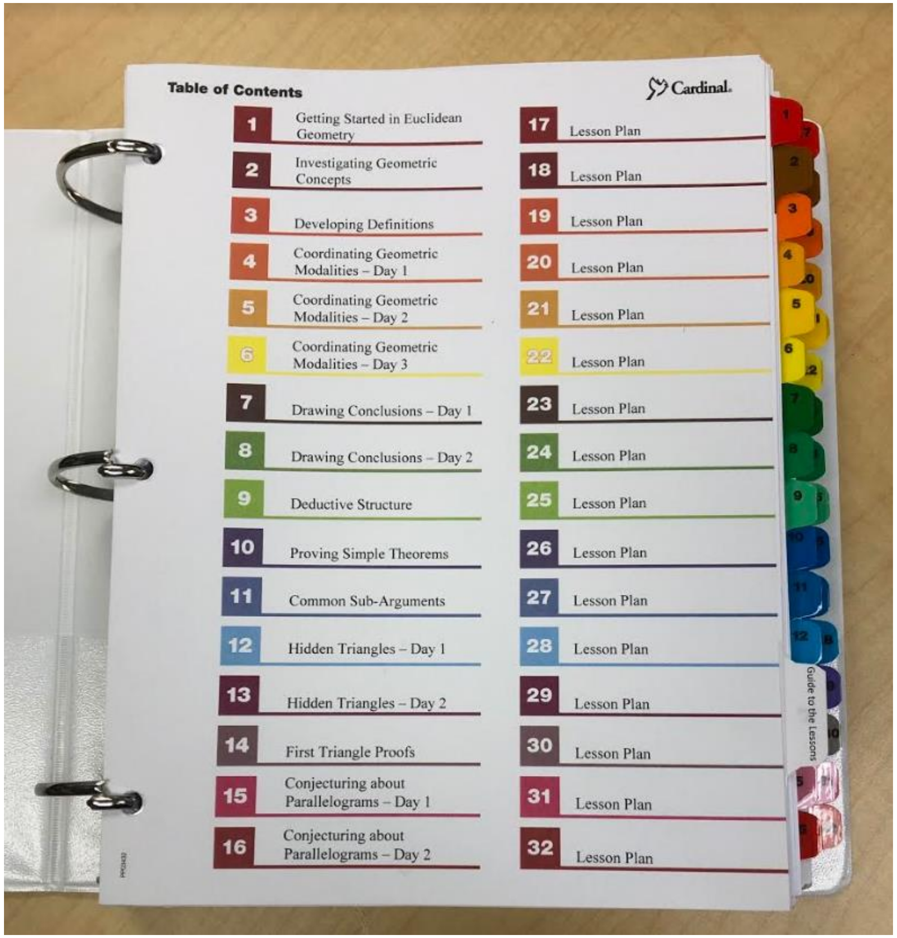
2016-2019

(Y2 - Y4)

Data Collected

Assessments	Interviews	Classroom Observations
1,550 Pre-Tests Administered (EGT) 1,278 Post-Tests Administered (SGT)	24 Teacher Interviews 31 Student Interviews	294 Classroom Observations

PISC Curriculum



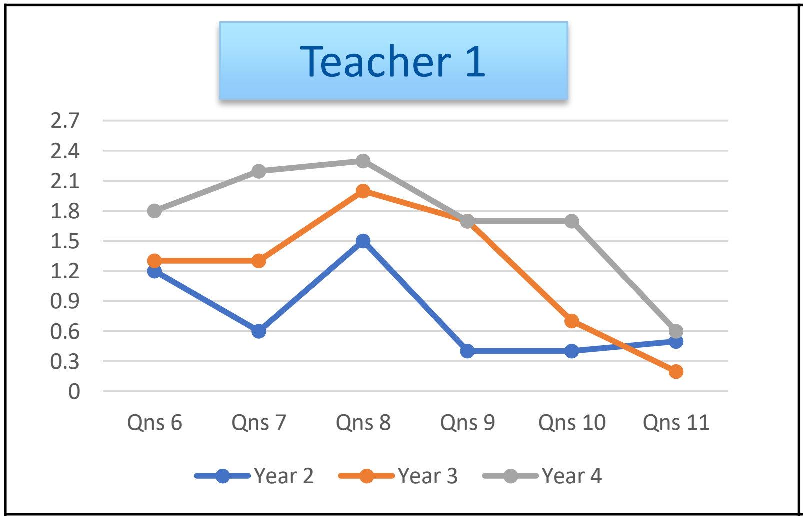
PISC Curriculum

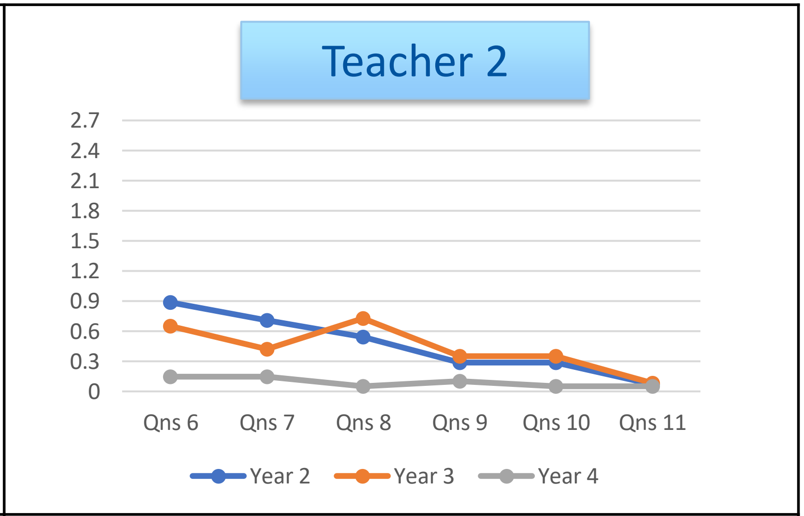
714 Pages

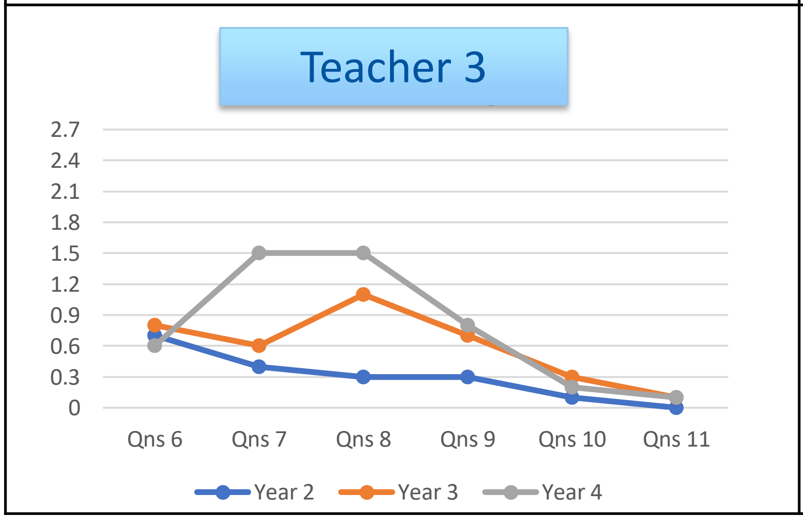
- 16 Detailed Lesson Plans
- Student Sheets
- Homework
- Extra Resources
- Answer Keys

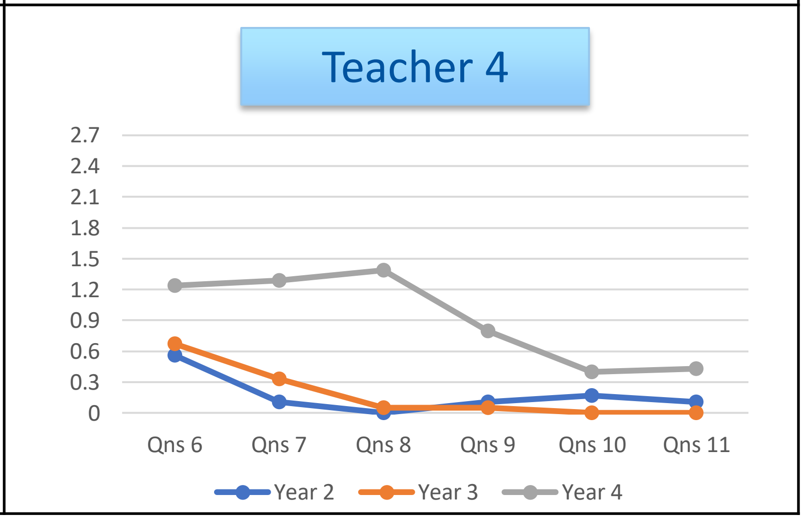
DID THE TREATMENT WORK?

Core Teachers Item Averages

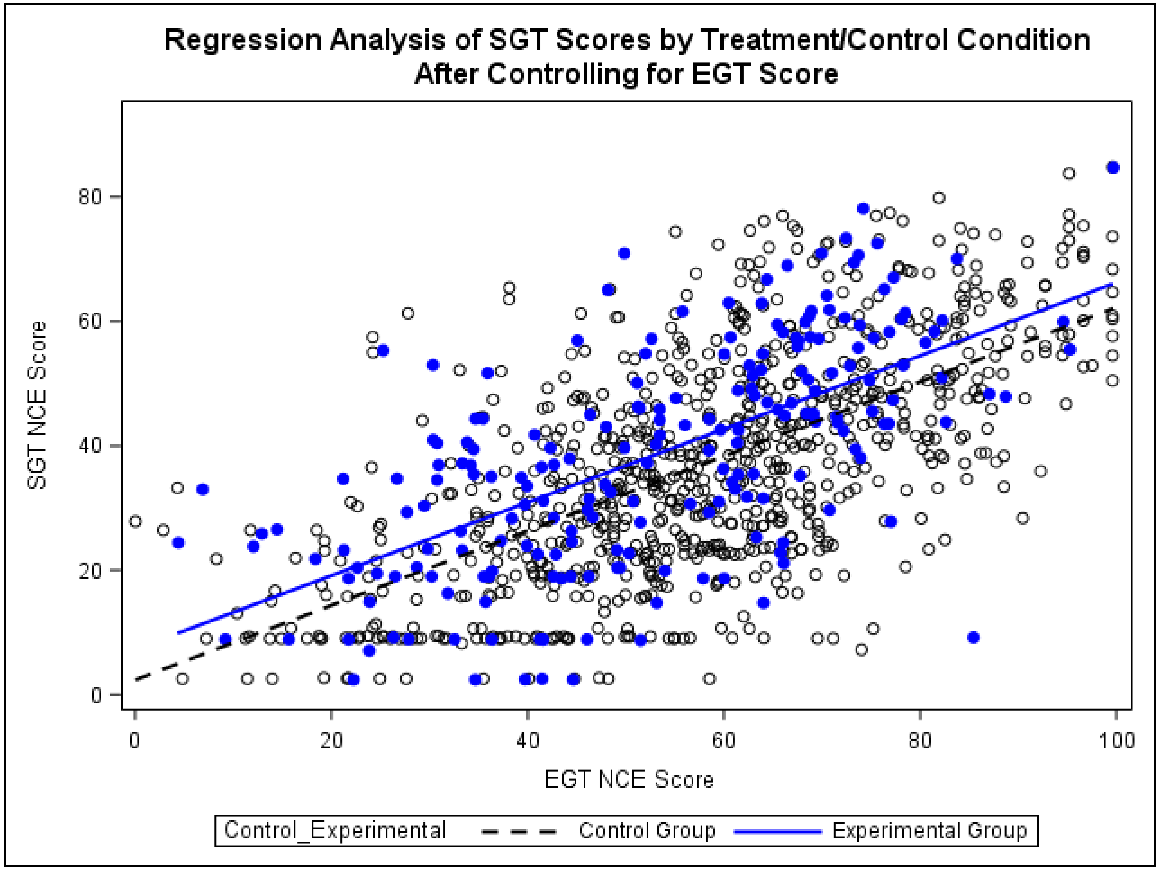








What is the estimated impact of the PISC curriculum on students' SGT scores?



What is the estimated impact of the PISC curriculum on students’ SGT scores (Year 1 vs. Year 3 only)?

HLM Model Parameters	Estimate	Standard Error	P-value
Fixed Effects			
Intercept	6.73	2.39	0.0125
EGT NCE Score	0.44	0.03	<.0001
8^{th} Grade Indicator	11.20	3.71	0.0026
CORE Treatment	6.61	1.75	0.0002
Random Effects (residuals)			
Teacher	45.22	18.0	0.0060
Student	122.34	6.47	<.0001

After controlling for grade level and EGT scores and restricting analyses to Year 1 and Year 3 data only, students in CORE classes scored 6.61 NCE points higher (ES = +.31 standard deviations) on the SGT ($p<.001$).

Gains made by students were significantly larger in classrooms using the PISC curriculum.

Michelle's Reflections

- "Collaboration between researchers and school personnel provides integrated perspectives for addressing critical issues in mathematics teaching and learning" (NCTM, 2012, p. 1).
- Impact of Attending to Student Thinking
- Cannot do this kind of work alone or on campus

Jen's Reflections

- Well worth the time and effort
- Confidence in content
- Better teaching overall

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Thank you!

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Email mcirillo@udel.edu for questions about or visit www.pisc.udel.edu for updates on the project.

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